

SOME NEW INTEGRAL INEQUALITIES FOR SEVERAL KINDS OF CONVEX FUNCTIONS

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ABSTRACT. In this study, we obtain some new integral inequalities for different classes of convex functions by using some elementary inequalities and classical inequalities like general Cauchy inequality and Minkowski inequality.

1. INTRODUCTION

The function $f : [a, b] \rightarrow \mathbb{R}$, is said to be convex, if we have

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$. This definition well-known in the literature and a huge amount of the researchers interested in this definition. We can define star-shaped functions on $[0, b]$ which satisfy the condition

$$f(tx) \leq tf(x)$$

for $t \in [0, 1]$.

The concept of m -convexity has been introduced by Toader in [5], an intermediate between the ordinary convexity and starshaped property, as following:

Definition 1. The function $f : [0, b] \rightarrow \mathbb{R}$, $b > 0$, is said to be m -convex, where $m \in [0, 1]$, if we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$. We say that f is m -concave if $-f$ is m -convex.

Several papers have been written on m -convex functions and we refer the papers [1], [2], [3], [7], [8] and [9].

In [4], Miheşan gave definition of (α, m) -convexity as following;

Definition 2. The function $f : [0, b] \rightarrow \mathbb{R}$, $b > 0$ is said to be (α, m) -convex, where $(\alpha, m) \in [0, 1]^2$, if we have

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

Denote by $K_m^\alpha(b)$ the class of all (α, m) -convex functions on $[0, b]$ for which $f(0) \leq 0$. If we choose $(\alpha, m) = (1, m)$, it can be easily seen that (α, m) -convexity reduces to m -convexity and for $(\alpha, m) = (1, 1)$, we have ordinary convex functions on $[0, b]$. In [6], Set *et al.* proved some inequalities related to (α, m) -convex functions.

Following definition of *log-convexity* is given by Pečarić *et. al.*

1991 *Mathematics Subject Classification.* 26D15.

Key words and phrases. convex functions, m -convex functions, s -convex functions, Minkowski Inequality, (α, m) -convex functions, general Cauchy inequality. *log-convex functions.*

Definition 3. A function $f : I \rightarrow [0, \infty)$ is said to be log-convex or multiplicatively convex if $\log t$ is convex, or, equivalently, if for all $x, y \in I$ and $t \in [0, 1]$ one has the inequality:

$$(1.1) \quad f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{1-t}.$$

The following inequality which well known in the literature as Minkowski Inequality is given as;

Let $p \geq 1$, $0 < \int_a^b f(x)^p dx < \infty$, and $0 < \int_a^b g(x)^p dx < \infty$. Then

$$(1.2) \quad \left(\int_a^b (f(x) + g(x))^p dx \right)^{\frac{1}{p}} \leq \left(\int_a^b f(x)^p dx \right)^{\frac{1}{p}} + \left(\int_a^b g(x)^p dx \right)^{\frac{1}{p}}.$$

The reverse of this inequality was given by Bougoffa in [16], as following;

Theorem 1. Let f and g be positive functions satisfying

$$0 < m \leq \frac{f(x)}{g(x)} \leq M, \quad \forall x \in [a, b].$$

Then

$$(1.3) \quad \left(\int_a^b f(x)^p dx \right)^{\frac{1}{p}} + \left(\int_a^b g(x)^p dx \right)^{\frac{1}{p}} \leq c \left(\int_a^b (f(x) + g(x))^p dx \right)^{\frac{1}{p}}.$$

where $c = \frac{M(m+1)+(M+1)}{(m+1)(M+1)}$.

Definition 4. [See [10]] Let $s \in (0, 1]$. A function $f : [0, \infty) \rightarrow [0, \infty)$ is said to be s -convex in the second sense if

$$(1.4) \quad f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in \mathbb{R}_+$ and $t \in [0, 1]$.

In [11], s -convexity introduced by Breckner as a generalization of convex functions. Also, Breckner proved the fact that the set valued map is s -convex only if the associated support function is s -convex function in [12]. Several properties of s -convexity in the first sense are discussed in the paper [10]. Obviously, s -convexity means just convexity when $s = 1$.

Theorem 2. [See [14]] Suppose that $f : [0, \infty) \rightarrow [0, \infty)$ is an s -convex function in the second sense, where $s \in (0, 1]$ and let $a, b \in [0, \infty)$, $a < b$. If $f \in L_1[0, 1]$, then the following inequalities hold:

$$(1.5) \quad 2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{s+1}.$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.5). The above inequalities are sharp.

Some new Hermite-Hadamard type inequalities based on concavity and s -convexity established by Kirmacı *et al.* in [15]. For related results see the papers [13], [14] and [15].

In this paper, we prove some inequalities for m -convex and s -convex and log-convex functions and we give some new inequalities for (α, m) -convex functions by using some classical inequalities and fairly elementary analysis.

2. MAIN RESULTS

We will start with the following Theorem which is involving m -convex functions.

Theorem 3. *Suppose that $f, g : [a, b] \rightarrow [0, \infty)$, $0 \leq a < b < \infty$, are m_1 -convex and m_2 -convex functions, respectively, where $m_1, m_2 \in (0, 1]$. If $f, g \in L_1[a, b]$, then the following inequality holds:*

$$(2.1) \quad \frac{1}{b-a} \int_a^b f^{\frac{x-a}{b-a}}(x) g^{\frac{b-x}{b-a}}(x) dx \leq \frac{1}{3} \left[f(b) + m_2 g\left(\frac{a}{m_2}\right) \right] + \frac{1}{6} \left[g(b) + m_1 f\left(\frac{a}{m_1}\right) \right].$$

Proof. From m_1 -convexity and m_2 -convexity of f and g respectively, we can write

$$f^t(tb + (1-t)a) \leq \left[tf(b) + m_1(1-t)f\left(\frac{a}{m_1}\right) \right]^t$$

and

$$g^{(1-t)}(tb + (1-t)a) \leq \left[tg(b) + m_2(1-t)g\left(\frac{a}{m_2}\right) \right]^{(1-t)}.$$

Since f, g are non-negative, we have

$$(2.2) \quad \begin{aligned} & f^t(tb + (1-t)a) g^{(1-t)}(tb + (1-t)a) \\ & \leq \left[tf(b) + m_1(1-t)f\left(\frac{a}{m_1}\right) \right]^t \left[tg(b) + m_2(1-t)g\left(\frac{a}{m_2}\right) \right]^{(1-t)}. \end{aligned}$$

Recall the General Cauchy Inequality (see [17], Theorem 3.1), let α and β be positive real numbers satisfying $\alpha + \beta = 1$. Then for every positive real numbers x and y , we always have

$$\alpha x + \beta y \geq x^\alpha y^\beta.$$

Applying the General Cauchy Inequality to the right hand side of (2.2), we get

$$\begin{aligned} & f^t(tb + (1-t)a) g^{(1-t)}(tb + (1-t)a) \\ & \leq t \left[tf(b) + m_1(1-t)f\left(\frac{a}{m_1}\right) \right] + (1-t) \left[tg(b) + m_2(1-t)g\left(\frac{a}{m_2}\right) \right]. \end{aligned}$$

By integrating with respect to t over $[0, 1]$, we have

$$\begin{aligned} & \int_0^1 f^t(tb + (1-t)a) g^{(1-t)}(tb + (1-t)a) dt \\ & \leq \frac{1}{3} \left[f(b) + m_2 g\left(\frac{a}{m_2}\right) \right] + \frac{1}{6} \left[g(b) + m_1 f\left(\frac{a}{m_1}\right) \right]. \end{aligned}$$

Hence, by taking into account the change of the variable $tb + (1-t)a = x$, $(b-a)dt = dx$, we obtain the required result. \square

Corollary 1. *If we choose $m_1 = m_2 = 1$ in Theorem 3, we have the inequality;*

$$\frac{1}{b-a} \int_a^b f^{\frac{x-a}{b-a}}(x) g^{\frac{b-x}{b-a}}(x) dx \leq \frac{1}{3} [f(b) + g(a)] + \frac{1}{6} [g(b) + f(a)].$$

Another result for m -convex functions is embodied in the following Theorem.

Theorem 4. *Suppose that $f, g : [0, b] \rightarrow \mathbb{R}$, $b > 0$, are m_1 -convex and m_2 -convex functions, respectively, where $m_1, m_2 \in (0, 1]$. If $f \in L_1[a, b]$, then the following inequality holds:*

$$\begin{aligned} (2.3) \quad & \frac{g(b)}{(b-a)^2} \int_a^b (x-a) f(x) dx + m_2 \frac{g\left(\frac{a}{m_2}\right)}{(b-a)^2} \int_a^b (b-x) f(x) dx \\ & + \frac{f(b)}{(b-a)^2} \int_a^b (x-a) g(x) dx + m_1 \frac{f\left(\frac{a}{m_1}\right)}{(b-a)^2} \int_a^b (b-x) g(x) dx \\ & \leq \frac{1}{b-a} \int_a^b f(x) g(x) dx + \frac{1}{3} f(b) g(b) + \frac{m_1}{6} f\left(\frac{a}{m_1}\right) g(b) \\ & + \frac{m_2}{6} f(b) g\left(\frac{a}{m_2}\right) + \frac{m_1 m_2}{3} f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right). \end{aligned}$$

Proof. Since f and g are m_1 -convex and m_2 -convex functions, respectively, we can write

$$f(tb + (1-t)a) \leq tf(b) + m_1(1-t)f\left(\frac{a}{m_1}\right)$$

and

$$g(tb + (1-t)a) \leq tg(b) + m_2(1-t)g\left(\frac{a}{m_2}\right).$$

By applying the elementary inequality, $e \leq f$ and $p \leq r$, then $er + fp \leq ep + fr$ for $e, f, p, r \in \mathbb{R}$, to above inequalities, we get:

$$\begin{aligned} & f(tb + (1-t)a) \left[tg(b) + m_2(1-t)g\left(\frac{a}{m_2}\right) \right] \\ & + g(tb + (1-t)a) \left[tf(b) + m_1(1-t)f\left(\frac{a}{m_1}\right) \right] \\ & \leq f(tb + (1-t)a) g(tb + (1-t)a) \\ & + \left[tg(b) + m_2(1-t)g\left(\frac{a}{m_2}\right) \right] \left[tf(b) + m_1(1-t)f\left(\frac{a}{m_1}\right) \right]. \end{aligned}$$

With a simple computation, we obtain

$$\begin{aligned}
& tf(tb + (1-t)a)g(b) + m_2(1-t)f(tb + (1-t)a)g\left(\frac{a}{m_2}\right) \\
& + tf(b)g(tb + (1-t)a) + m_1(1-t)f\left(\frac{a}{m_1}\right)g(tb + (1-t)a) \\
\leq & f(tb + (1-t)a)g(tb + (1-t)a) + t^2f(b)g(b) + m_1t(1-t)f\left(\frac{a}{m_1}\right)g(b) \\
& + m_2t(1-t)f(b)g\left(\frac{a}{m_2}\right) + m_1m_2(1-t)^2f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right).
\end{aligned}$$

By integrating this inequality with respect to t over $[0, 1]$ and by using the change of the variable $tb + (1-t)a = x$, $(b-a)dt = dx$, the proof is completed. \square

Corollary 2. *If we choose $m_1 = m_2 = 1$ in Theorem 4, we have the inequality;*

$$\begin{aligned}
& \frac{g(b)}{(b-a)^2} \int_a^b (x-a)f(x)dx + \frac{g(a)}{(b-a)^2} \int_a^b (b-x)f(x)dx \\
& + \frac{f(b)}{(b-a)^2} \int_a^b (x-a)g(x)dx + \frac{f(a)}{(b-a)^2} \int_a^b (b-x)g(x)dx \\
\leq & \frac{1}{b-a} \int_a^b f(x)g(x)dx + \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b)
\end{aligned}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$ and $N(a,b) = f(a)g(b) + f(b)g(a)$.

Corollary 3. *If we choose the functions f, g as increasing functions in Corollary 2, we obtain the following result:*

$$\begin{aligned}
& \frac{g(a)}{(b-a)^2} \left[\int_a^b (x-a)f(x)dx + \int_a^b (b-x)f(x)dx \right] \\
& + \frac{f(a)}{(b-a)^2} \left[\int_a^b (x-a)g(x)dx + \int_a^b (b-x)g(x)dx \right] \\
\leq & \frac{1}{b-a} \int_a^b f(x)g(x)dx + \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b).
\end{aligned}$$

Corollary 4. *If we chose $m_1 = m_2 = 1$ and $g(x) = 1$ in Theorem 4, we have the inequality;*

$$\begin{aligned} & \frac{1}{(b-a)^2} \left[\int_a^b (x-a)f(x) dx + \int_a^b (b-x)f(x) dx \right] \\ & + \frac{f(b)}{(b-a)^2} \int_a^b (x-a)dx + \frac{f(a)}{(b-a)^2} \int_a^b (b-x)dx \\ & \leq \frac{1}{b-a} \int_a^b f(x) dx + \frac{f(a) + f(b)}{2}. \end{aligned}$$

Following inequality also holds for m -convex functions.

Theorem 5. *Suppose that $f, g : [a, b] \rightarrow [0, \infty)$, $0 \leq a < b < \infty$, are m_1 -convex and m_2 -convex functions, respectively, where $m_1, m_2 \in (0, 1]$. If $f, g \in L_1[a, b]$ and f, g satisfy following condition*

$$0 < m \leq \frac{f(x)}{g(x)} \leq M, \quad \forall x \in [a, b]$$

then the following inequality holds:

$$\begin{aligned} & \frac{1}{c} \left[\left(\int_a^b f(x)^p dx \right)^{\frac{1}{p}} + \left(\int_a^b g(x)^p dx \right)^{\frac{1}{p}} \right] \\ & \leq \left(\frac{2^{p-1}(b-a)}{p+1} \right)^{\frac{1}{p}} \left([f(b) + g(b)]^p - \left[m_1 f\left(\frac{a}{m_1}\right) + m_2 g\left(\frac{a}{m_2}\right) \right]^p \right)^{\frac{1}{p}} \end{aligned}$$

where $c = \frac{M(m+1)+(M+1)}{(m+1)(M+1)}$ and $p \geq 1$.

Proof. Since f and g are m_1 -convex and m_2 -convex functions, respectively, we can write

$$(2.4) \quad f(tb + (1-t)a) \leq tf(b) + m_1(1-t)f\left(\frac{a}{m_1}\right)$$

and

$$(2.5) \quad g(tb + (1-t)a) \leq tg(b) + m_2(1-t)g\left(\frac{a}{m_2}\right).$$

By adding (2.4) and (2.5), we get

$$\begin{aligned} f(tb + (1-t)a) + g(tb + (1-t)a) & \leq tf(b) + m_1(1-t)f\left(\frac{a}{m_1}\right) \\ & + tg(b) + m_2(1-t)g\left(\frac{a}{m_2}\right). \end{aligned} \quad (2.6)$$

For $p \geq 1$, taking p -th power of both sides of the inequality (2.6) and by using the elementary inequality, $(e + f)^p \leq 2^{p-1}(e^p + f^p)$ where $e, f \in \mathbb{R}$, then we get

$$\begin{aligned} & [f(tb + (1-t)a) + g(tb + (1-t)a)]^p \\ & \leq 2^{p-1} \left(t^p [f(b) + g(b)]^p + (1-t)^p \left[m_1 f\left(\frac{a}{m_1}\right) + m_2 g\left(\frac{a}{m_2}\right) \right]^p \right). \end{aligned}$$

Integrating with respect to t over $[0, 1]$ and by using the change of the variable $tb + (1 - t)a = x$ and $(b - a)dt = dx$, we obtain

$$(2.7) \quad \frac{1}{b-a} \int_a^b (f(x) + g(x))^p dx \leq \frac{2^{p-1}}{p+1} \left([f(b) + g(b)]^p - \left[m_1 f\left(\frac{a}{m_1}\right) + m_2 g\left(\frac{a}{m_2}\right) \right]^p \right).$$

By taking $\frac{1}{p}$ -th power of both sides of the inequality (2.7) and by using the inequality (1.3), we get the desired inequality. Which completes the proof. \square

Corollary 5. *Under the assumptions of Theorem 5, if we choose $m_1 = m_2 = 1$ and take the limit of both sides as $p \rightarrow 1$, we obtain the following inequality:*

$$\int_a^b [f(x) + g(x)] dx \leq \left(\frac{c(b-a)}{2} \right) [[f(b) + g(b)] - [(f(a) + g(a))]]$$

We will give an inequality for s -convex functions in the following theorem. In the next theorem we will also make use of the Beta function of Euler type, which is for $x, y > 0$ defined

as

$$\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

Theorem 6. *Suppose that $f, g : [0, \infty) \rightarrow [0, \infty)$ are s_1 -convex and s_2 -convex functions in the second sense, respectively, where $s_1, s_2 \in [0, 1]$. Then the following inequality holds:*

$$\begin{aligned} \frac{1}{b-a} \int_a^b f^{\frac{x-a}{b-a}}(x) g^{\frac{b-x}{b-a}}(x) dx &\leq \frac{1}{s_1+2} f(b) + \beta(2, s_1+1) f(a) \\ &\quad + \frac{1}{s_2+2} g(b) + \beta(2, s_2+1) g(a). \end{aligned}$$

Proof. Since f and g are s_1 -convex and s_2 -convex functions, respectively, we can write

$$f^t(tb + (1-t)a) \leq [t^{s_1} f(b) + (1-t)^{s_1} f(a)]^t$$

and

$$g^{(1-t)}(tb + (1-t)a) \leq [t^{s_2} g(b) + (1-t)^{s_2} g(a)]^{(1-t)}.$$

Since f, g are non-negative, we have

$$(2.8) \quad \begin{aligned} &f^t(tb + (1-t)a) g^{(1-t)}(tb + (1-t)a) \\ &\leq [t^{s_1} f(b) + (1-t)^{s_1} f(a)]^t [t^{s_2} g(b) + (1-t)^{s_2} g(a)]^{(1-t)}. \end{aligned}$$

By using the General Cauchy Inequality in (2.8), we get

$$\begin{aligned} &f^t(tb + (1-t)a) g^{(1-t)}(tb + (1-t)a) \\ &\leq t[t^{s_1} f(b) + (1-t)^{s_1} f(a)] + (1-t)[t^{s_2} g(b) + (1-t)^{s_2} g(a)]. \end{aligned}$$

By integrating with respect to t over $[0, 1]$, we have

$$\begin{aligned} & \int_0^1 f^t(tb + (1-t)a) g^{(1-t)}(tb + (1-t)a) dt \\ & \leq \int_0^1 [t^{s_1+1} f(b) + t(1-t)^{s_1} f(a) + t^{s_2+1} g(b) + t(1-t)^{s_2} g(a)] dt. \end{aligned}$$

Hence, by taking into account the change of the variable $tb + (1-t)a = x$, $(b-a)dt = dx$, we obtain the required result. \square

Corollary 6. *If we choose $s_1 = s_2 = 1$ in Theorem 6, we have the inequality;*

$$\frac{1}{b-a} \int_a^b f^{\frac{x-a}{b-a}}(x) g^{\frac{b-x}{b-a}}(x) dx \leq \frac{1}{3} [f(b) + g(b)] + \frac{1}{6} [f(a) + g(a)].$$

Theorem 7. *Let f, g be s -convex functions in the second sense and $\alpha + \beta = 1$ then the following inequality holds:*

$$\frac{1}{b-a} \int_a^b f^\alpha(x) \cdot g^\beta(x) dx \leq \frac{1}{s+1} [\alpha [f(a) + f(b)] + \beta [g(a) + g(b)]].$$

Proof. If we use the general Cauchy inequality with s -convexity of f and g we get:

$$\begin{aligned} & f^\alpha(ta + (1-t)b) g^\beta(ta + (1-t)b) \\ & \leq \alpha f(ta + (1-t)b) + \beta g(ta + (1-t)b) \\ & \leq \alpha [t^s f(a) + (1-t)^s f(b)] + \beta [t^s g(a) + (1-t)^s g(b)] \end{aligned}$$

By integrating with respect to t over $[0, 1]$, we have

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f^\alpha(ta + (1-t)b) g^\beta(ta + (1-t)b) dt \\ & \leq \alpha \int_a^b [t^s f(a) + (1-t)^s f(b)] dt + \beta \int_a^b [t^s g(a) + (1-t)^s g(b)] dt \\ & = \frac{1}{s+1} [\alpha [f(a) + f(b)] + \beta [g(a) + g(b)]]. \end{aligned}$$

With the change of variable $ta + (1-t)b = x$ we obtain:

$$\frac{1}{b-a} \int_a^b f^\alpha(x) g^\beta(x) dx \leq \frac{1}{s+1} [\alpha [f(a) + f(b)] + \beta [g(a) + g(b)]].$$

That is the desired result.

A similar result for \log -convex functions is as follows: \square

Theorem 8. Let f, g be log-convex functions and $\alpha + \beta = 1$ where L denotes the logarithmic mean then the following inequality holds:

$$\frac{1}{b-a} \int_a^b f^\alpha(x) g^\beta(x) dx \leq \alpha L[f(a), f(b)] + \beta L[g(a), g(b)].$$

Logarithmic Mean: $L(a, b) = \frac{a-b}{\log(a)-\log(b)}$ where $a, b \in \mathbb{R}^+$.

Proof. If we use the general Cauchy inequality with log-convexity of f and g we get:

$$\begin{aligned} & f^\alpha(ta + (1-t)b) g^\beta(ta + (1-t)b) \\ & \leq \alpha f(ta + (1-t)b) + \beta g(ta + (1-t)b) \\ & \leq \alpha [f(a)]^t [f(b)]^{(1-t)} + \beta [g(a)]^t [g(b)]^{(1-t)} \end{aligned}$$

By integrating both sides with respect to t over $[0, 1]$, we have

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f^\alpha(ta + (1-t)b) g^\beta(ta + (1-t)b) dt \\ & \leq \alpha \int_a^b [f(a)]^t [f(b)]^{(1-t)} dt + \beta \int_a^b [g(a)]^t [g(b)]^{(1-t)} dt \\ & = \alpha \frac{f(a) - f(b)}{\log[f(a)] - \log[f(b)]} + \beta \frac{g(a) - g(b)}{\log[g(a)] - \log[g(b)]} \\ & = \alpha L[f(a), f(b)] + \beta L[g(a), g(b)]. \end{aligned}$$

With the change of variable $ta + (1-t)b = x$ we obtain the desired result.

In following two Theorems we obtain results for (α, m) -convex functions: \square

Theorem 9. Suppose that $f, g : [a, b] \rightarrow [0, \infty)$, $0 \leq a < b < \infty$, are (α_1, m_1) -convex and (α_2, m_2) -convex functions, respectively, where $\alpha_1, m_1, \alpha_2, m_2 \in (0, 1]$. If $f, g \in L_1[a, b]$, then the following inequality holds:

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f^{\frac{x-a}{b-a}}(x) g^{\frac{b-x}{b-a}}(x) dx \\ & \leq \frac{1}{\alpha_1 + 2} f(b) + \frac{m_1}{2(\alpha_1 + 2)} f\left(\frac{a}{m_1}\right) \\ & \quad + \frac{1}{(\alpha_2 + 1)(\alpha_2 + 2)} g(b) + \frac{m_2(\alpha_2^2 + 3\alpha_2)}{2(\alpha_2 + 1)(\alpha_2 + 2)} g\left(\frac{a}{m_2}\right). \end{aligned}$$

Proof. Since f, g are (α_1, m_1) -convex and (α_2, m_2) -convex functions, respectively, we can write

$$f^t(tb + (1-t)a) \leq \left[t^{\alpha_1} f(b) + m_1(1-t^{\alpha_1}) f\left(\frac{a}{m_1}\right) \right]^t$$

and

$$g^{(1-t)}(tb + (1-t)a) \leq \left[t^{\alpha_2} g(b) + m_2(1-t^{\alpha_2}) g\left(\frac{a}{m_2}\right) \right]^{(1-t)}.$$

Since f, g are non-negative, we have

$$(2.9) \quad f^t(tb + (1-t)a)g^{(1-t)}(tb + (1-t)a) \\ \leq \left[t^{\alpha_1}f(b) + m_1(1-t^{\alpha_1})f\left(\frac{a}{m_1}\right) \right]^t \left[t^{\alpha_2}g(b) + m_2(1-t^{\alpha_2})g\left(\frac{a}{m_2}\right) \right]^{(1-t)}.$$

By using the General Cauchy Inequality in (2.9), we get

$$f^t(tb + (1-t)a)g^{(1-t)}(tb + (1-t)a) \\ \leq t \left[t^{\alpha_1}f(b) + m_1(1-t^{\alpha_1})f\left(\frac{a}{m_1}\right) \right] + (1-t) \left[t^{\alpha_2}g(b) + m_2(1-t^{\alpha_2})g\left(\frac{a}{m_2}\right) \right].$$

By integrating with respect to t over $[0, 1]$, we have

$$\int_0^1 f^t(tb + (1-t)a)g^{(1-t)}(tb + (1-t)a)dt \\ \leq \frac{1}{\alpha_1 + 2}f(b) + \frac{m_1}{2(\alpha_1 + 2)}f\left(\frac{a}{m_1}\right) \\ + \frac{1}{(\alpha_2 + 1)(\alpha_2 + 2)}g(b) + \frac{m_2(\alpha_2^2 + 3\alpha_2)}{2(\alpha_2 + 1)(\alpha_2 + 2)}g\left(\frac{a}{m_2}\right).$$

Hence, by taking into account the change of the variable $tb + (1-t)a = x$, $(b-a)dt = dx$, we obtain the required result. \square

Corollary 7. *If we choose $\alpha_1 = \alpha_2 = 1$ in Theorem 9, we have the inequality (2.1).*

Theorem 10. *Suppose that $f, g : [a, b] \rightarrow [0, \infty)$, $0 \leq a < b < \infty$, are (α_1, m_1) -convex and (α_2, m_2) -convex functions, respectively, where $\alpha_1, m_1, \alpha_2, m_2 \in (0, 1]$. If $f, g \in L_1[a, b]$, then the following inequality holds:*

$$\frac{g(b)}{(b-a)^{\alpha_2+1}} \int_a^b (x-a)^{\alpha_2} f(x) dx + m_2 \frac{g\left(\frac{a}{m_2}\right)}{(b-a)^{\alpha_2+1}} \int_a^b [(b-a)^{\alpha_2} - (x-a)^{\alpha_2}] f(x) dx \\ + \frac{f(b)}{(b-a)^{\alpha_1+1}} \int_a^b (x-a)^{\alpha_1} g(x) dx + m_1 \frac{f\left(\frac{a}{m_1}\right)}{(b-a)^{\alpha_1+1}} \int_a^b [(b-a)^{\alpha_1} - (x-a)^{\alpha_1}] g(x) dx \\ \leq \frac{1}{b-a} \int_a^b f(x)g(x) dx + \frac{1}{\alpha_1 + \alpha_2 + 1} f(b)g(b) + \frac{m_2\alpha_2}{(\alpha_1 + 1)(\alpha_1 + \alpha_2 + 1)} g\left(\frac{a}{m_2}\right) f(b) \\ + \frac{m_1\alpha_1}{(\alpha_1 + 1)(\alpha_1 + \alpha_2 + 1)} f\left(\frac{a}{m_1}\right) g(b) + \frac{m_1m_2\alpha_1\alpha_2(\alpha_1 + \alpha_2 + 2)}{(\alpha_1 + 1)(\alpha_2 + 1)(\alpha_1 + \alpha_2 + 1)} f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right).$$

Proof. Since f, g are (α_1, m_1) -convex and (α_2, m_2) -convex functions, respectively, we can write

$$f(tb + (1-t)a) \leq t^{\alpha_1}f(b) + m_1(1-t^{\alpha_1})f\left(\frac{a}{m_1}\right)$$

and

$$g(tb + (1-t)a) \leq t^{\alpha_2}g(b) + m_2(1-t^{\alpha_2})g\left(\frac{a}{m_2}\right).$$

By using the elementary inequality, $e \leq f$ and $p \leq r$, then $er + fp \leq ep + fr$ for $e, f, p, r \in \mathbb{R}$ and by a similar argument to the proof of Theorem 4, we get the required result. \square

Corollary 8. *If we choose $\alpha_1 = \alpha_2 = 1$ in Theorem 10, we have the inequality (2.3).*

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